

# Factorising numbers with a Bose–Einstein condensate

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The problem to express a natural number  $N$  as a product of natural numbers without regard to order corresponds to a thermally isolated non-interacting Bose gas in a one-dimensional potential with logarithmic energy eigenvalues. This correspondence is used for characterising the probability distribution which governs the number of factors in a randomly selected factorisation of an asymptotically large  $N$ . Asymptotic upper bounds on both the skewness and the excess of this distribution, and on the total number of factorisations, are conjectured. The asymptotic formulas are checked against exact numerical data obtained with the help of recursion relations. It is also demonstrated that for large numbers which are the product of different primes the probability distribution approaches a Gaussian, while identical prime factors give rise to non-Gaussian statistics.

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## I. INTRODUCTION

Each integer number  $N$  can be written in a unique way as the product of prime numbers  $p_i$  with integer multiplicities  $n_i$ ,

$$N = \prod_{i=1}^m p_i^{n_i} . \quad (1)$$

Every other possibility to decompose  $N$  into integer factors larger than 1 is obtained by multiplying out some of its prime factors. For  $N = 30$ , for instance, we have the factorisations

$$\begin{aligned} 30 &= 2 \cdot 3 \cdot 5 \\ &= 5 \cdot 6 \\ &= 3 \cdot 10 \\ &= 2 \cdot 15 \\ &= 30 . \end{aligned} \quad (2)$$

It is understood that the ordering of the factors does not matter here, so that  $5 \cdot 6$  is not distinguished from  $6 \cdot 5$ . Let  $\Phi(N, k)$  denote the number of such factorisations of  $N$  which contain exactly  $k$  factors, and let  $\Omega(N)$  be the total number of factorisations. Then, according to the above list,  $\Phi(30, 1) = 1$ ,  $\Phi(30, 2) = 3$ , and  $\Phi(30, 3) = 1$ , giving a total of  $\Omega(30) = 5$  factorisations.

It is obvious that the number  $\Omega(N)$  of possible factorisations of  $N$  does not only depend on the total number of prime factors of  $N$ , but also on the multiplicities with which the different prime factors occur: Taking  $N = 12$ , we have

$$\begin{aligned} 12 &= 2 \cdot 2 \cdot 3 \\ &= 2 \cdot 6 \\ &= 3 \cdot 4 \\ &= 12 , \end{aligned} \quad (3)$$

so that  $\Phi(12, 1) = 1$ ,  $\Phi(12, 2) = 2$ , and  $\Phi(12, 3) = 1$ , adding up to  $\Omega(12) = 4$ . Since two of the three prime factors of  $N = 12$  are equal, the number of different combinations of prime factors is less than for  $N = 30$ , which possesses three different prime factors.

We therefore divide the natural numbers into equivalence classes: Two numbers  $N_1$  and  $N_2$  are said to be equivalent if they give rise to the same pattern of factorisations, meaning that  $\Phi(N_1, k) = \Phi(N_2, k)$  for all  $k$ . Particular equivalence classes consist of numbers which are some power of a single prime  $p$ : If  $N = p^m$ , or

$$\ln N = m \ln p \quad (4)$$

for some integer  $m$ , the task of factorising  $N$  is equivalent to the task of partitioning the exponent  $m$  into integer summands, *i.e.*, to the famous number-partitioning problem *partitio numerorum* introduced by Euler<sup>1</sup>. This problem, which in itself plays a significant role in several areas of modern mathematics<sup>2,3,4</sup>, is connected to a number of topics occurring in statistical physics, ranging from lattice animals<sup>5,6</sup> and combinatorial optimisation<sup>7</sup> over Fermion-Boson transmutation<sup>8</sup> to quantum entropy and energy currents flowing in a one-dimensional channel connecting thermal reservoirs<sup>9</sup>.

In the general case, taking the logarithm of the prime factor decomposition (1) yields

$$\ln N = \sum_{i=1}^m n_i \ln p_i. \quad (5)$$

Seen from the viewpoint of statistical physics, this latter equation allows for an interesting interpretation: If we consider an ideal Bose gas consisting of sufficiently many particles, confined such that the single-particle energies  $\varepsilon_\nu$ , when suitably made dimensionless, are given by the logarithms of the primes,  $\varepsilon_\nu = \ln p_\nu$  with  $p_\nu = 1, 2, 3, 5, 7, 11, \dots$  for  $\nu = 0, 1, 2, 3, 4, 5, \dots$ , the equation (5) indicates one particular microstate of the system, where the total energy  $\ln N$  is distributed among the particles in such a way that  $n_i$  is the occupation number of the energy level  $\varepsilon_i$ . Additional particles which are not excited remain in the ground state, forming a Bose–Einstein condensate. Such surplus particles do not contribute to the energy, since  $\varepsilon_0 = \ln 1 = 0$ . Ideal Bose gases with a single-particle spectrum corresponding to the sequence of the logarithms of the prime numbers have recently been studied by Tran and Bhaduri<sup>10</sup>, with particular emphasis placed on differences between the fluctuation of the number of condensate particles in different statistical ensembles.

A few remarks concerning low-dimensional, ideal Bose–Einstein condensates might appear in order. The example of an ideal Bose gas of  $M$  particles in a one-dimensional harmonic potential with oscillator frequency  $\omega$  captures the essentials: If one considers the usual thermodynamic limit, meaning  $M \rightarrow \infty$  and  $\omega \rightarrow 0$  such that the product  $M\omega$  remains constant, the Bose–Einstein condensation temperature approaches zero<sup>11</sup>. However, this is not the relevant limit here; we rather have to consider the limit of large particle numbers at *constant* oscillator frequency. In this case there still exists a nonzero characteristic temperature  $T_C$  for the onset of a macroscopically large occupation of the ground state, roughly given by  $T_C = (\hbar\omega/k_B)M/\ln(M)$ , which approaches infinity for  $M \rightarrow \infty$ <sup>12,13</sup>. Thus, there exists a condensate for any finite temperature, if the particle number is sufficiently large.

Since we wish to treat *all* possible factorisations (not only those into primes) of a given number  $N$  without regard to the order of the factors, the Bose-gas analogy requires the single-particle spectrum

$$\varepsilon_\nu = \ln(\nu + 1), \quad \nu = 0, 1, 2, \dots \quad (6)$$

We point out that such a spectrum might actually be realisable: As shown in the appendix A, within the quasi-classical approximation the eigenvalues of a particle in a one-dimensional logarithmic potential  $V(x) = V_0 \ln(|x|/L)$  are given by  $V_0 \ln(2\nu + 1)$ , up to a constant; the restriction to odd numbers  $2\nu + 1$  is not essential.

The key point here is that the analogy between the factorisation problem and an ideal Bose gas with logarithmic single-particle spectrum allows us to invoke well-established methods from statistical physics for obtaining information on number-theoretical properties of large composite integers. In this paper, we focus on the probability distribution

$$P_N(k) = \frac{\Phi(N, k)}{\Omega(N)} \quad (7)$$

for given (asymptotically) large integers  $N$ , *i.e.*, on the probability of finding  $k$  factors in a randomly selected factorisation of a large  $N$ . We will proceed as follows: In section II we state recursion relations required for the numerical evaluation of the exact quantities  $\Phi(N, k)$ , deferring the derivation to appendix B. We then briefly explain in section III the method used to obtain, by means of a detour from the microcanonical to the canonical ensemble and back, asymptotic expressions for the cumulants of the distributions (7). In the following, we focus on the two extreme kinds of equivalence classes, namely those made up of powers of a single prime on the one hand, and those consisting of products of different primes on the other, and state the respective asymptotic formulas for the cumulants in sections III and IV. In this way, an interesting feature will become apparent: While the presence of identical prime factors in the first case introduces Bose-like correlations which prevent the distributions (7) from becoming Gaussian even in the asymptotic limit, large products of different primes do indeed lead to almost Gaussian distributions. The paper closes with a brief summary in section V.

In passing, we point out that the problem considered in this paper should be clearly distinguished from a similar problem known as *factorisatio numerorum*, first investigated by Kalmár<sup>14</sup>. In this latter connection one counts all *ordered* sequences  $(n_1, n_2, \dots, n_k)$  of integers  $n_1, n_2, \dots, n_k \geq 2$  which yield  $N$  when multiplied,  $n_1 \cdot n_2 \cdot \dots \cdot n_k = N$ .

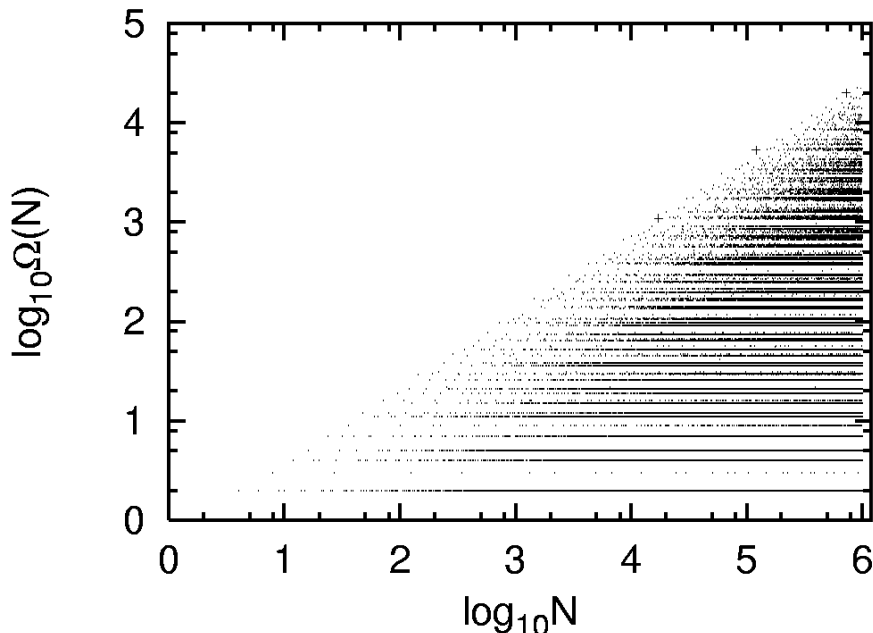


FIG. 1: The number of possibilities  $\Omega(N)$  to factorise an integer  $N$  into products of natural numbers, for  $1 \leq N \leq 10^6$ . For each  $N$ , the exact value  $\Omega(N)$  has been computed with the help of the recursion relation (9). While there are lots of numbers which are products of two primes, giving  $\Omega(N) = 2$ , only relatively few numbers can be written as the third power of a prime, implying  $\Omega(N) = 3$ . Composite numbers like  $17280 = 2^7 \cdot 3^3 \cdot 5$ ,  $120960 = 2^7 \cdot 3^3 \cdot 5 \cdot 7$ , or  $725760 = 2^8 \cdot 3^4 \cdot 5 \cdot 7$  yield rather high values of  $\Omega(N)$ , indicated by crosses.

Denoting the number of such ordered sequences by  $a_N$  (with  $a_1 = 1$ ), one deduces  $\sum_{N \geq 1} a_N N^{-s} = [2 - \zeta(s)]^{-1}$ , where  $\zeta(s)$  is Riemann's zeta function; the task then is to find the asymptotic behaviour of the sum function

$$A(x) \equiv \sum_{1 \leq N \leq x} a_N. \quad (8)$$

Recent results, and further information on this problem, have been collected in ref.<sup>15</sup>. In contrast, in the present paper we do *not* count different orderings of factors as different configurations, or microstates. It is precisely this identification of different orderings which leads to the Bose-gas analogy (with Bosons, the question “which particle occupies which state” is meaningless), and thus opens up the avenue followed here.

## II. RECURSION RELATIONS

In order to determine the exact distributions (7), we employ a recursion relation: Let  $\Gamma_k(N)$  be the number of factorisations of  $N$  into  $k$  or less factors. Then, as shown in appendix B, one has

$$\Gamma_k(N) = \frac{1}{k} \sum_{n=1}^k \sum_{\substack{\nu=1 \\ N \bmod \nu^n = 0}}^N \Gamma_{k-n}(N/\nu^n), \quad (9)$$

with  $\Gamma_k(1) = 1$  and  $\Gamma_0(N > 1) = 0$ .

Contributions to the second sum in equation (9) arise only when  $\nu^n$  divides  $N$ . Starting from  $\Gamma_1(N) = 1$ , the sequence  $\Gamma_k(N)$  increases with increasing  $k$  (unless  $N$  happens to be a prime) until it reaches its final value  $\Omega(N)$ , since  $N$  cannot be expressed as a product of more than  $\log_2(N)$  integer factors greater than 1:

$$\Omega(N) = \Gamma_k(N), \quad k \geq \log_2(N). \quad (10)$$

Figure 1 shows a logarithmic plot of  $\Omega(N)$  for  $1 \leq N \leq 10^6$ . Each data point  $[N, \Omega(N)]$  has been marked by an individual dot; the equivalence classes clearly manifest themselves as horizontal lines. While these data might suggest

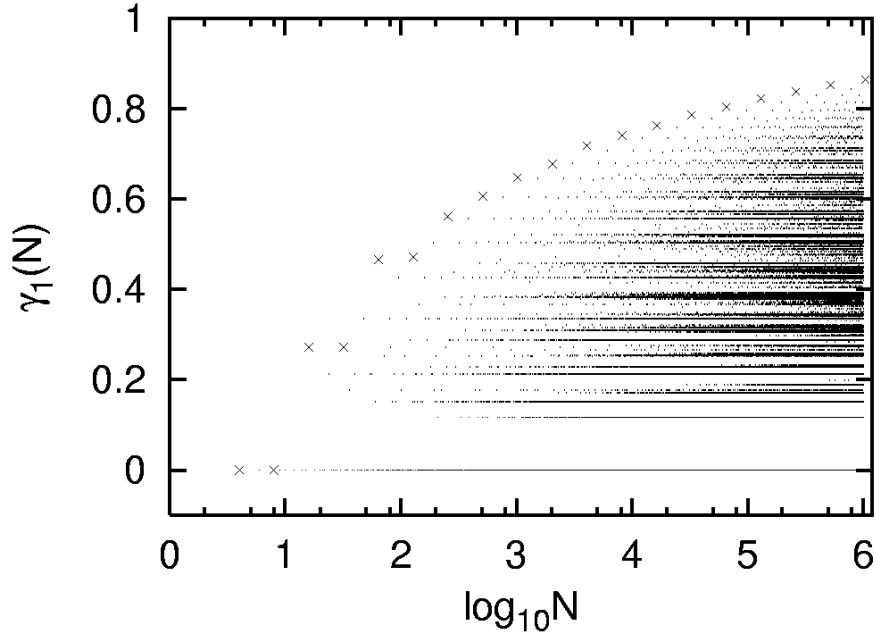


FIG. 2: Exact skewness  $\gamma_1(N)$ , as defined in equation (13), for the probability distributions (7), with  $1 \leq N \leq 10^6$ . There are some numbers  $N$  for which the distribution is symmetric, so that  $\gamma_1(N) = 0$ . In contrast, numbers which are integer powers of 2 give rise to a particularly large skewness, as indicated by the crosses.

an upper bound on  $\Omega(N)$  on the order of  $N^{0.77}$ , the reader should be warned that the true asymptotics are not reached for  $N$  as small as  $10^6$ ; a correct upper bound will be stated later.

The number of possibilities to factorise  $N$  into exactly  $k$  factors is now given by

$$\Phi(N, k) = \Gamma_k(N) - \Gamma_{k-1}(N), \quad (11)$$

so that we have access to the probability distribution (7). As in previous investigations of ground-state fluctuations of non-interacting and weakly interacting Bose gases<sup>16,17,18</sup>, it is useful to characterise such a distribution in terms of its cumulants  $\kappa^{(\ell)}$ <sup>19</sup>, which directly quantify its deviation from a Gaussian: For a Gaussian distribution,  $\kappa^{(\ell)} = 0$  for  $\ell > 2$ . The lowest four cumulants are related to the mean value  $\langle k \rangle$  and the central moments  $\mu^{(\ell)} = \langle (k - \langle k \rangle)^\ell \rangle$  through the relations<sup>19</sup>

$$\begin{aligned} \kappa^{(1)} &= \langle k \rangle, \\ \kappa^{(2)} &= \mu^{(2)}, \\ \kappa^{(3)} &= \mu^{(3)}, \\ \kappa^{(4)} &= \mu^{(4)} - 3 \left( \mu^{(2)} \right)^2. \end{aligned} \quad (12)$$

Normalising the third and the fourth cumulant we respect to the second, one obtains the skewness

$$\gamma_1 \equiv \frac{\kappa^{(3)}}{(\kappa^{(2)})^{3/2}}, \quad (13)$$

which characterises the asymmetry of the underlying probability distribution, and the excess

$$\gamma_2 \equiv \frac{\kappa^{(4)}}{(\kappa^{(2)})^2}, \quad (14)$$

which characterises its flatness.

In figure 2 we display exact data for the skewness  $\gamma_1(N)$  of the distributions (7), revealing that their asymmetry becomes particularly large when  $N = 2^m$ . It is clear that all numbers within the same equivalence class yield the same cumulants, and thus the same values of  $\gamma_1$  and  $\gamma_2$ .

Since the general algorithm (9) requires rather a large amount of computer memory, it pays to focus on particular equivalence classes: For numbers  $N$  which are powers of a prime  $p$ ,

$$N = p^m, \quad (15)$$

one has

$$\Phi(p^m, k) = \sum_{\nu=1}^{\min\{m-k, k\}} \Phi(p^{m-k}, \nu); \quad (16)$$

this recursion relation still has to be solved numerically. If, on the other hand,  $N$  is a product of distinct primes,

$$N = \prod_{i=1}^m p_i, \quad p_i \neq p_j \text{ for } i \neq j, \quad (17)$$

the probability distribution is given by the relation

$$\Phi\left(\prod_{j=1}^m p_j, k\right) = \Phi\left(\prod_{j=1}^{m-1} p_j, k-1\right) + k\Phi\left(\prod_{j=1}^{m-1} p_j, k\right) \quad (18)$$

with  $\Phi(p_1, 1) = 1$  and  $\Phi\left(\prod_{j=1}^{m-1} p_j, m\right) = 0$ . As explained in appendix C, this relation even admits a closed solution:

$$\Phi\left(\prod_{j=1}^m p_j, k\right) = \frac{(-1)^k}{k!} \sum_{\ell=1}^k (-1)^\ell \binom{k}{\ell} \ell^m. \quad (19)$$

### III. ASYMPTOTIC FORMULAS FOR FACTORISING POWERS OF A PRIME

The problem of finding all factorisations of a given number  $N$  corresponds, within the Bose-gas analogy, to a problem of microcanonical statistics: Given the total energy  $\ln N$ , the task is to find all possible microstates, *i.e.*, all possibilities for distributing the energy over the available single-particle levels  $\varepsilon_\nu$ . Particles which are not excited and thus remain in the condensate contribute with  $\ln \varepsilon_0 = 0$  to the energy, or with a factor of 1 to the product. Hence, the picture to have in mind is that of a partially condensed Bose gas, with the excited particles corresponding to factors larger than 1, and a sufficiently large supply of condensed particles in the ground state.

While the *microcanonical* requirement to keep the total energy constant introduces severe technical difficulties, the statistics of partially condensed ideal Bose gases become much simpler in the *canonical* ensemble, where the system is kept in contact with a thermal reservoir of temperature  $T$ . It is precisely at this point that the Bose-gas picture becomes essential: One can obtain information about the microcanonical number factorisation problem by invoking the notion of temperature.

We employ the first energy gap  $\varepsilon_1 - \varepsilon_0$  to introduce the dimensionless inverse temperature

$$b = (\varepsilon_1 - \varepsilon_0)\beta, \quad (20)$$

where  $\beta = 1/(k_B T)$ , as usual. A decisive role for the canonical statistics of a partially condensed, ideal Bose gas is played by the grand canonical partition function  $\Xi_{\text{ex}}(b, z)$  of an auxiliary system from which the single-particle ground state  $\varepsilon_0$  has been removed, while all other levels  $\varepsilon_\nu$  remain unaltered. If there is an infinite supply of condensed particles, in the spirit of the so-called Maxwell's demon ensemble<sup>20</sup>, this function has the exact integral representation<sup>21,22</sup>

$$\ln \Xi_{\text{ex}}(b, z) = \frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} dt b^{-t} \Gamma(t) \eta(t) g_{t+1}(z), \quad (21)$$

where

$$g_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\alpha} \quad (22)$$

denotes the familiar Bose functions<sup>23</sup>, and

$$\eta(t) \equiv \sum_{\nu=1}^{\infty} \left( \frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_\nu - \varepsilon_0} \right)^t \quad (23)$$

is a series determined by the single-particle spectrum. The real number  $\tau > 0$  in equation (21) is chosen such that all poles of the integrand lie on the left hand side of the path of integration. The value of this representation (21), and of the subsequent representation (25), lies in the fact that it lends itself to the derivation of an asymptotic series which captures the low- $b$ -behaviour, if one collects, from right to left on the real axis, the residues of the respective integrands. With respect to an actual Bose gas, the underlying assumption of an infinite number of ground-state particles restricts the validity of equation (21) to temperatures low enough to guarantee the existence of a condensate. In the case of the number factorisation problem there is no similar restriction, owing to the fact that multiplying any product by an arbitrary amount of factors of unity does not change its value: In the number-theoretic context the representation (21) is *exact*.

The  $\ell$ -th cumulants  $\kappa_{\text{cn}}^{(\ell)}(b)$  of the canonical distribution governing the number of excited particles in the gas are then immediately obtained by differentiation,

$$\kappa_{\text{cn}}^{(\ell)}(b) = \left( z \frac{\partial}{\partial z} \right)^\ell \ln \Xi_{\text{ex}}(b, z) \Big|_{z=1}, \quad (24)$$

giving<sup>21,22</sup>

$$\kappa_{\text{cn}}^{(\ell)}(b) = \frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} dt \, b^{-t} \Gamma(t) \eta(t) \zeta(t+1-\ell), \quad (25)$$

where  $\zeta(t) = \sum_{n=1}^{\infty} n^{-t}$  is the Riemann zeta function. We will only consider non-interacting Bosons in trapping potentials with single particle energy levels  $\varepsilon_\nu$  ( $\nu = 0, 1, 2 \dots$ ) such that the series (23) converges for  $t > t_0$ , with some real  $t_0 > 0$ .

The strategy of employing the canonical ensemble for solving a microcanonical problem hinges on the possibility to get rid of the temperature in a second step, and to find an expression for the microcanonical cumulants. This is done with the help of the saddle-point method<sup>20</sup>: The “energy-temperature” relation reads

$$n+1 = - \frac{\partial}{\partial b} \ln \Xi_{\text{ex}}(b, z) \Big|_{b=b_0(z)}, \quad (26)$$

where  $n$  denotes the energy in units of  $\varepsilon_1 - \varepsilon_0$ , and  $b_0(z)$  is the saddle point. The generating function of the microcanonical cumulants then takes the form<sup>24</sup>

$$\begin{aligned} \ln Y(n, z) &= \ln \Xi_{\text{ex}}(b_0(z), z) + n b_0(z) - \frac{1}{2} \ln 2\pi \\ &\quad - \frac{1}{2} \ln \left( \frac{\partial^2}{\partial b^2} \ln \Xi_{\text{ex}}(b, z) \right) \Big|_{b=b_0(z)}, \end{aligned} \quad (27)$$

and the desired microcanonical cumulants are finally calculated by taking derivatives,

$$\kappa_{\text{mc}}^{(\ell)}(n) = \left( z \frac{d}{dz} \right)^\ell \ln Y(n, z) \Big|_{z=1}. \quad (28)$$

The method sketched here requires knowledge about the analytical properties of the function  $\eta(t)$  introduced in equation (23). For applications to the number factorisation problem, the single-particle energies are determined by the equivalence class of the number  $N$ : In the simplest case (15), the possible single-particle energies are integer multiples of  $\ln p$ , so that  $\eta(t)$  coincides with the Riemann zeta function  $\zeta(t)$ , and the factorisation of  $N$  is equivalent to the partition of  $m = \log_p N$  into summands. The evaluation of the above formalism for the partition problem has already been discussed in detail<sup>24</sup>, so we merely quote the results: For numbers of the type (15), the asymptotic

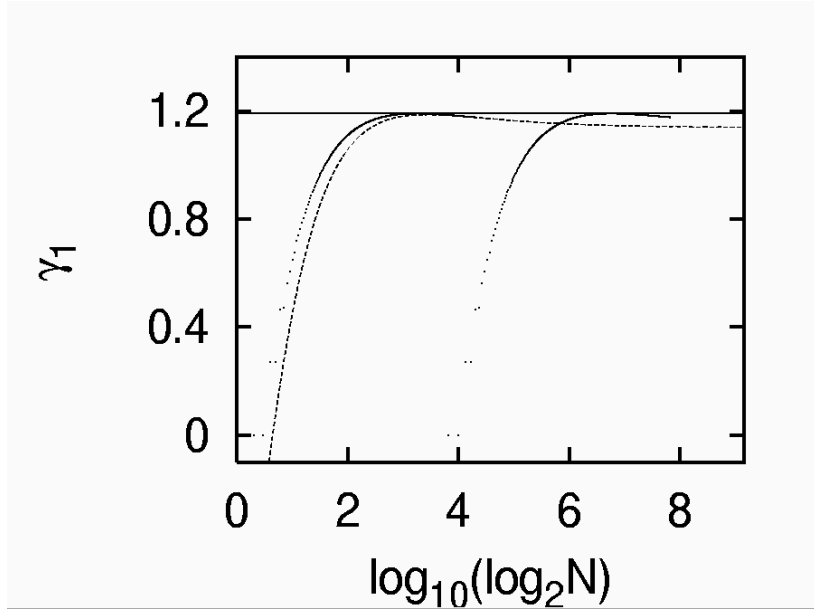


FIG. 3: Skewness (13) for large numbers  $N = 2^m$  and  $N = (10^{1000} + 453)^m$ , as obtained from the recursion relation (16) (dots). The dashes correspond to the evaluation of the asymptotic formula (29) for  $N = 2^m$ ; the horizontal line indicates the maximum value  $\gamma_{1, \max} \simeq 1.1907$ .

formula for the skewness (13) reads

$$\begin{aligned} \gamma_1 = & \frac{12\sqrt{6}\zeta(3)}{\pi^3} \\ & + \frac{1}{\sqrt{\log_p(N)}} \left[ 0.10128 [\ln(\log_p(N))]^2 - 0.37376 \ln(\log_p(N)) - 1.7078 \right] \\ & + \frac{1}{\log_p(N)} \left[ 0.0075008 [\ln(\log_p(N))]^4 + 0.025681 [\ln(\log_p(N))]^3 \right. \\ & + 0.020024 [\ln(\log_p(N))]^2 - 0.23028 \ln(\log_p(N)) - 0.56984 \left. \right] \\ & + \mathcal{O}(\log_p(N)^{-3/2}), \end{aligned} \quad (29)$$

while the excess (14) takes the form

$$\begin{aligned} \gamma_2 = & 2.4 \\ & + \frac{1}{\sqrt{\log_p(N)}} \left[ 0.28440 [\ln(\log_p(N))]^2 - 0.56714 \ln(\log_p(N)) - 10.064 \right] \\ & + \frac{1}{\log_p(N)} \left[ 0.025276 [\ln(\log_p(N))]^4 + 0.022329 [\ln(\log_p(N))]^3 \right. \\ & - 0.33809 [\ln(\log_p(N))]^2 + 0.73538 \ln(\log_p(N)) + 3.7863 \left. \right] \\ & + \mathcal{O}(\log_p(N)^{-3/2}). \end{aligned} \quad (30)$$

In order to demonstrate the accuracy of these formulas, we compare in figures 3 and 4 exact values for the skewness and the excess, corresponding to large numbers  $N = p^m$ , with the predictions of the equations (29) and (30). The agreement is excellent. It is also visible that both  $\gamma_1(N)$  and  $\gamma_2(N)$  approach constant values only for numbers  $N$  of the order  $p^{(10^{10})}$ . The finding that the limiting values of skewness and excess are nonzero expresses the fact that for numbers of the form (15) the distribution (7) remains non-Gaussian even for asymptotically large  $N$ .

The exact numerical data further reveal that both skewness and excess reach their limiting values from above; the skewness adopts its maximum value

$$\gamma_{1, \max} \simeq 1.1907 \quad (31)$$

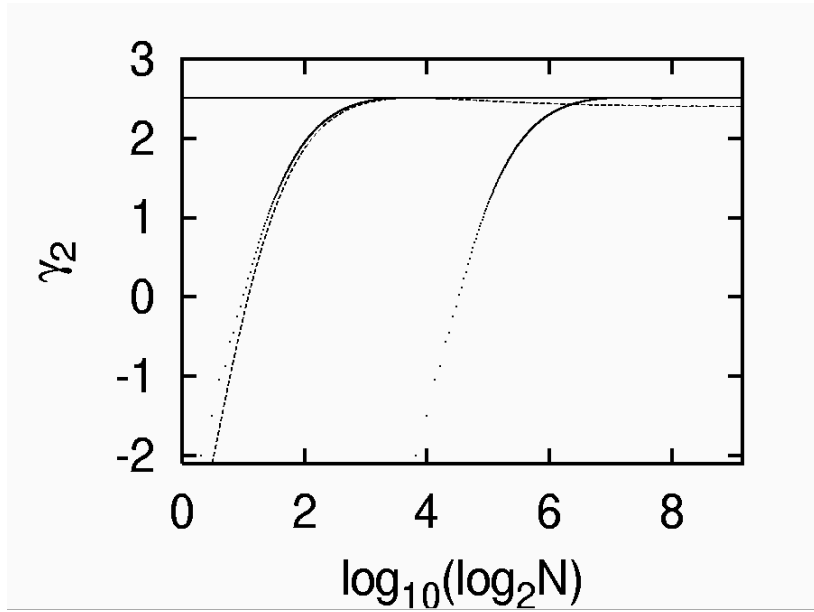


FIG. 4: Excess (14) for large numbers  $N = 2^m$  and  $N = (10^{1000} + 453)^m$ , as obtained from the recursion relation (16) (dots). The dashes correspond to the evaluation of the asymptotic formula (30) for  $N = 2^m$ ; the horizontal line indicates the maximum value  $\gamma_{2, \max} \simeq 2.5120$ .

for  $N = p^{1730}$ . The maximum of the excess,

$$\gamma_{2, \max} \simeq 2.5120 \quad (32)$$

lies at  $N = p^{5507}$ .

#### IV. ASYMPTOTIC FORMULAS FOR FACTORISING PRODUCTS OF DIFFERENT PRIMES

We now turn to the equivalence classes consisting of products of different primes, and restrict ourselves to numbers  $N$  which are the product of the first  $m$  primes,

$$N = \prod_{i=1}^m p_i, \quad (33)$$

where  $m$  is large. In this case, the single-particle energies accessible to the Bose gas are given by

$$\varepsilon_{\{\nu_i\}_{i=1\dots m}} = \sum_{i=1}^m \nu_i \ln(p_i), \quad \nu_i \in \{0, 1\}. \quad (34)$$

The restriction  $\nu_i \in \{0, 1\}$  stems from the fact that each prime  $p_i$  does either contribute to a given factor of  $N$ , or it does not. The task then is to characterise the analytical properties of the series  $\eta(t)$  corresponding to this spectrum. We circumvent this severe difficulty by considering a simpler, but combinatorically equivalent problem: We replace the actual energy levels (34) by

$$\varepsilon_{\{\nu_i\}_{i=1\dots m}} = \sum_{i=1}^m \nu_i a^{i-1}, \quad \nu_i \in \{0, 1\}, \quad (35)$$

where  $a > 1$  is a transcendental number which has to assure the “prime” character of  $\exp(a^m)$  in the sense that it be not possible to express  $a^m$  as a sum of the form  $\sum_{i=1}^{m-1} n_i a^i$ , for arbitrary integers  $n_i$ , since a prime cannot be represented as a product of arbitrary powers of lower primes, and any  $m$ , since the function  $\eta(t)$  provided by the substitute (35) acquires a contribution for each  $m$ . This requirement forces us to take the limit  $a \rightarrow 1$  in the end:



Otherwise, there would be an infinite sequence of integers  $m$  for which the forbidden equality  $a^m = \sum_{i=1}^{m-1} n_i a^i$  could be satisfied with arbitrary accuracy. It is also clear that the total energy  $\sum_{i=1}^m \ln(p_i)$  has to be replaced by

$$n \equiv \sum_{i=1}^m a^{i-1} . \quad (36)$$

With this surrogate, we now have

$$\eta(t) = \sum_{m=0}^{\infty} \sum_{\left\{ \begin{smallmatrix} \{\nu_i\}_{i=1\dots m+1} \\ \nu_{m+1}=1 \end{smallmatrix} \right\}} \frac{1}{(\varepsilon_{\{\nu_i\}})^t} , \quad (37)$$

and easily obtain upper and lower bounds on the second sum, assuming  $t > 0$ :

$$\frac{2^m}{(a^m)^t} \geq \sum_{\left\{ \begin{smallmatrix} \{\nu_i\}_{i=1\dots m+1} \\ \nu_{m+1}=1 \end{smallmatrix} \right\}} \frac{1}{(\varepsilon_{\{\nu_i\}})^t} \geq \frac{2^m}{\left(\frac{a^{m+1}-1}{a-1}\right)^t} > \frac{2^m}{\left(\frac{a^{m+1}}{a-1}\right)^t} . \quad (38)$$

Thus, for  $t > \frac{\ln(2)}{\ln(a)}$  we can perform the sum over  $m$  and obtain

$$\frac{1}{1 - \frac{2}{a^t}} > \eta(t) > \left(\frac{a-1}{a}\right)^t \frac{1}{1 - \frac{2}{a^t}} , \quad (39)$$

implying that  $\eta(t)$  has a simple pole at

$$c \equiv \frac{\ln(2)}{\ln(a)} , \quad (40)$$

with a residuum  $r$  which can at least be estimated,

$$\frac{1}{\ln a} > r > \frac{\left(\frac{a-1}{a}\right)^{\frac{\ln(2)}{\ln(a)}}}{\ln a} . \quad (41)$$

These informations suffice for an approximate evaluation of the integral representation (25) of the canonical cumulants. We are interested in their asymptotic behaviour for small  $b$ , as defined in equation (20): While the temperature has to be low enough to guarantee the existence of a Bose condensate,  $k_B T = 1/\beta$  has to remain large in comparison with  $\varepsilon_1 - \varepsilon_0$ , so that sufficiently many states above the ground state remain populated. Now the small- $b$ -asymptotics of the cumulants (25) are determined by the rightmost pole of the respective integrand. Since  $\Gamma(t)$  has simple poles for  $t = 0, -1, -2, \dots$ , and  $\zeta(t+1-\ell)$  has a simple pole at  $t = \ell$ , the dominant pole is provided by  $\eta(t)$  at  $t = c$ : Taking  $a$  close to unity, as required, results in a value of  $c$  larger than any fixed, finite number.

Restricting ourselves to this dominant pole, the residue theorem then yields

$$\kappa_{\text{cn}}^{(\ell)}(b) \sim \frac{\Gamma(c) \zeta(c+1-\ell) r}{b^c} , \quad \ell = 0, 1, 2, \dots , \quad (42)$$

with the  $\sim$ -sign indicating asymptotic equality. It follows immediately that

$$\lim_{b \rightarrow 0} \frac{\kappa_{\text{cn}}^{(\ell)}(b)}{\left(\kappa_{\text{cn}}^{(2)}(b)\right)^{\frac{\ell}{2}}} = 0 , \quad \ell > 2 , \quad (43)$$

which means that the canonical distribution becomes Gaussian-like for small temperatures.

Next, one needs the energy-temperature relation (26) in order to return to the microcanonical ensemble. Since, according to equation (24),  $\ln \Xi_{\text{ex}}(b, 1)$  coincides with  $\kappa_{\text{cn}}^{(0)}(b)$ , to leading order this relation takes the form

$$b(n) \sim \left( \frac{c \Gamma(c) \zeta(c+1) r}{n+1} \right)^{\frac{1}{c+1}} , \quad (44)$$

and the leading-order term of the microcanonical cumulants (28) is given by

$$\kappa_{\text{mc}}^{(\ell)}(n) \sim \frac{\Gamma(c) \zeta(c+1-\ell) r}{[b(n)]^c} . \quad (45)$$

Utilising

$$\frac{\Gamma(c+1)^{1/(c+1)}}{c} \longrightarrow \exp(-1) \quad (46)$$

for  $c \rightarrow \infty$ , we then find

$$\kappa_{\text{mc}}^{(\ell)}(n) \sim \exp(-1)(n+1) , \quad \ell \geq 1 . \quad (47)$$

As an immediate consequence, we have

$$\lim_{n \rightarrow \infty} \frac{\kappa_{\text{mc}}^{(\ell)}(n)}{\left(\kappa_{\text{mc}}^{(2)}(n)\right)^{\frac{\ell}{2}}} = 0 , \quad \ell > 2 , \quad (48)$$

so that the approach to a Gaussian is also met in the microcanonical ensemble.

We now have to get rid of the auxiliary energy-like quantity  $n$ , and to re-introduce the number to be factorised,  $N$ . By virtue of the definition (36) one has  $n \sim m$  for  $a \rightarrow 1$ , meaning that the “energy” approaches the number of prime factors of  $N$  when  $N$  becomes large. Hence, we find

$$\gamma_1 \sim \frac{1}{\sqrt{\exp(-1)(m+1)}} \quad (49)$$

and

$$\gamma_2 \sim \frac{1}{\exp(-1)(m+1)} , \quad (50)$$

so that skewness and excess have been expressed in terms of the number  $m$  of primes contained in the product (33), provided  $m$  is sufficiently large. The final links in our chain of arguments are then provided by two results from analytic number theory: Firstly, if  $N$  equals the product of the first  $m$  primes from 2 to  $p_m$ , as in our case (33), one has<sup>25</sup>

$$\ln N \sim p_m . \quad (51)$$

Secondly, the number  $\pi(p_m)$  of primes less than  $p_m$ , which is  $m$ , is asymptotically given by<sup>25</sup>

$$\pi(p_m) \sim \frac{p_m}{\ln(p_m)} . \quad (52)$$

These estimates combine to yield

$$m \sim \frac{\ln(N)}{\ln(\ln(N))} . \quad (53)$$

Thus, the final asymptotic expressions for the skewness and the excess of the distributions (7) pertaining to large numbers  $N$  expressible as a product of all primes up to some  $p_m$  become

$$\gamma_1 \sim \frac{1}{\sqrt{\exp(-1) \left( \frac{\ln(N)}{\ln(\ln(N))} + 1 \right)}} \quad (54)$$

and

$$\gamma_2 \sim \frac{1}{\exp(-1) \left( \frac{\ln(N)}{\ln(\ln(N))} + 1 \right)} . \quad (55)$$

Again, we check these results against exact numerical calculations: The figures 5 and 6 depict skewness and excess of the probability distributions (7) corresponding to large numbers  $N$  of the type (33), again contrasting exact data points with the predictions of the asymptotic formulas. For smaller  $N$ , the agreement is not quite as good as in the previous figures 3 and 4. This can be attributed to the fact that equations (54) and (55) merely stem from a leading-order analysis, as necessitated by the rather complicated function  $\eta(t)$  underlying the  $N$  studied here. Nonetheless, the approach to the Gaussian limits  $\gamma_1 = 0$  and  $\gamma_2 = 0$  is captured correctly.

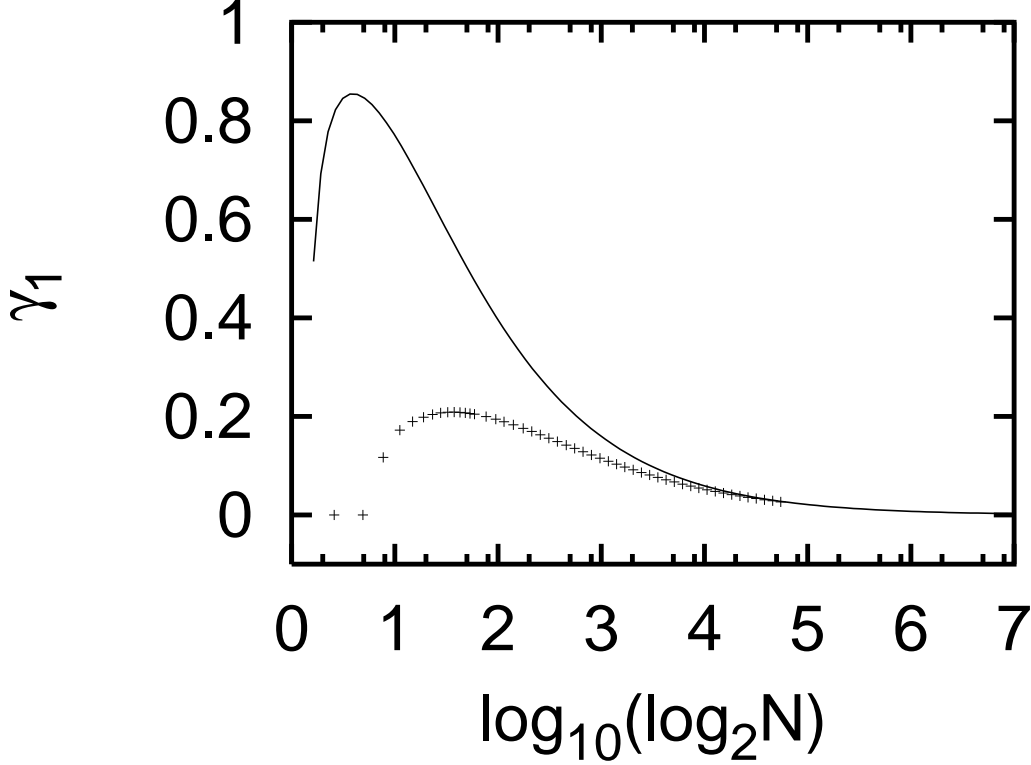


FIG. 5: Skewness (13) for large numbers  $N$  which are the product of the first  $m$  primes. Crosses indicate exact numerical data, obtained with the recursion relation (18); the full line corresponds to the asymptotic expression (54).

## V. DISCUSSION

We have used the correspondence between an ideal Bose gas with logarithmic single-particle levels and the number factorisation problem to characterise the “number-of-factors”-distribution (7) for large integers  $N$ . The properties of this distribution depend on the equivalence class of  $N$ , that is, on the multiplicity of its prime factors. The case (15), with all the prime factors being equal, constitutes one extreme; we have shown that for numbers of this type the skewness (13) and the excess (14) asymptotically approach the limiting values  $\gamma_{1,\infty} = 12\sqrt{6}\zeta(3)/\pi^3 \simeq 1.1395$  and  $\gamma_{2,\infty} = 2.4$ . We conjecture that for *any* number  $N$ , the skewness  $\gamma_1(N)$  and the excess  $\gamma_2(N)$  do not exceed the maximum values (31) and (32) adopted by numbers of this particular type:

**Conjecture 1** *For every integer  $N$ , the skewness (13) of the probability distribution (7) governing the number of factors in a randomly selected product decomposition is bounded from above by  $\gamma_{1,\max} \simeq 1.1906570491$ , while its excess (14) is bounded by  $\gamma_{2,\max} \simeq 2.5119565935$ .*

Moreover, for a given total number of prime factors (which equals  $\sum_{i=1}^m n_i$  for  $N = \prod_{i=1}^m p_i^{n_i}$ ), the total number of factorisations  $\Omega(N)$  becomes largest for integers  $N$  of the type (17), for which all prime factors differ from each other. Upper and lower bounds on the number of factorisations for integers of this particular type have been derived in appendix C. While the lower bound clearly pertains only to the equivalence classes considered there, we conjecture that the upper bound inferred from the inequality (C11) holds for *all* large integers. Again utilising the asymptotic relation (53), we thus formulate

**Conjecture 2** *For large integers  $N$ , an upper bound on the number  $\Omega(N)$  of possible factorisations is provided by*

$$\begin{aligned} \ln[\Omega(N)] &< \frac{\ln(N)}{\ln(\ln(N))} \ln \left[ \frac{\ln(N)}{\ln(\ln(N))} \right] \\ &\sim \ln(N). \end{aligned} \tag{56}$$

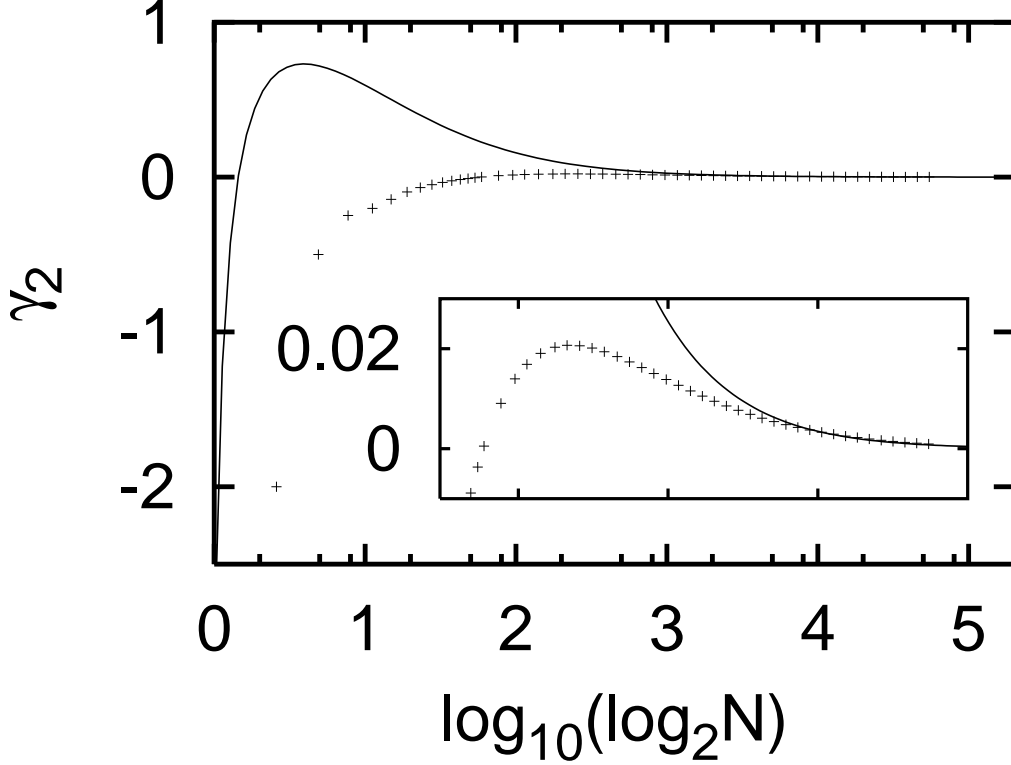


FIG. 6: Excess (14) for large numbers  $N$  which are the product of the first  $m$  primes. Crosses indicate exact numerical data, obtained with the recursion relation (18); the full line corresponds to the asymptotic expression (55). The inset demonstrates that this expression actually describes the asymptotics correctly.

In figure 7 we display the bounds (C11) for products of different primes, together with exact numerical data. Despite the somewhat crude approximations, these bounds describe the data quite well. Hence, when  $\ln(\ln(N)) \gg 1$ , the number  $\Omega(N)$  grows as  $N$  at least for some integers  $N$ , which was not obvious at all from the limited data collected in figure 1.

The actual content of conjecture 2 lies in the circumstance that it might still be possible to trade equality of prime factors for smallness of  $N$ : Constructing some composite integer by choosing more than one prime factor equal to  $p_1 = 2$ , say, certainly reduces the value of  $\Omega$  below the one that is attained when all prime factors are different, but also reduces the value of  $N$  itself. In this way, one might try and maintain a relatively high value of  $\Omega(N)$ , while minimising  $N$ . Indeed, the three examples singled out by the crosses in figure 1 indicate that in some cases such a procedure might lead to data points which fall at least close to the envelope of all pairs  $[N, \Omega(N)]$ . However, we conjecture that the bound (56) holds nonetheless.

For powers of a single prime  $p$ , the factorisation problem is equivalent to Euler's number partitioning problem<sup>1</sup>, so that the asymptotics can be deduced from the Hardy-Ramanujan formula<sup>2</sup> for the number of partitions:

$$\ln[\Omega(N)] \sim \pi \sqrt{\frac{2}{3} \log_p N} \quad \text{for } N = p^m. \quad (57)$$

As expected, in this case  $\ln[\Omega(N)]$  lies well below the conjectured upper bound (56). On the other hand, a strict upper bound can be established as follows:  $N$  cannot be the product of more than  $\log_2(N)$  primes. Assuming that these are all different (which can only overestimate the number of factors), and again using the result (C11), one finds

$$\ln[\Omega(N)] < \log_2(N) \ln[\log_2(N)] \quad \text{for all } N. \quad (58)$$

Our conjectured bound (56) is clearly stronger than this “safe” one.

Besides these number-theoretical insights made possible by the close correspondence between the microcanonical statistics of an ideal Bose gas and the factorisation problem, there also is a conceptual aspect of our work: For

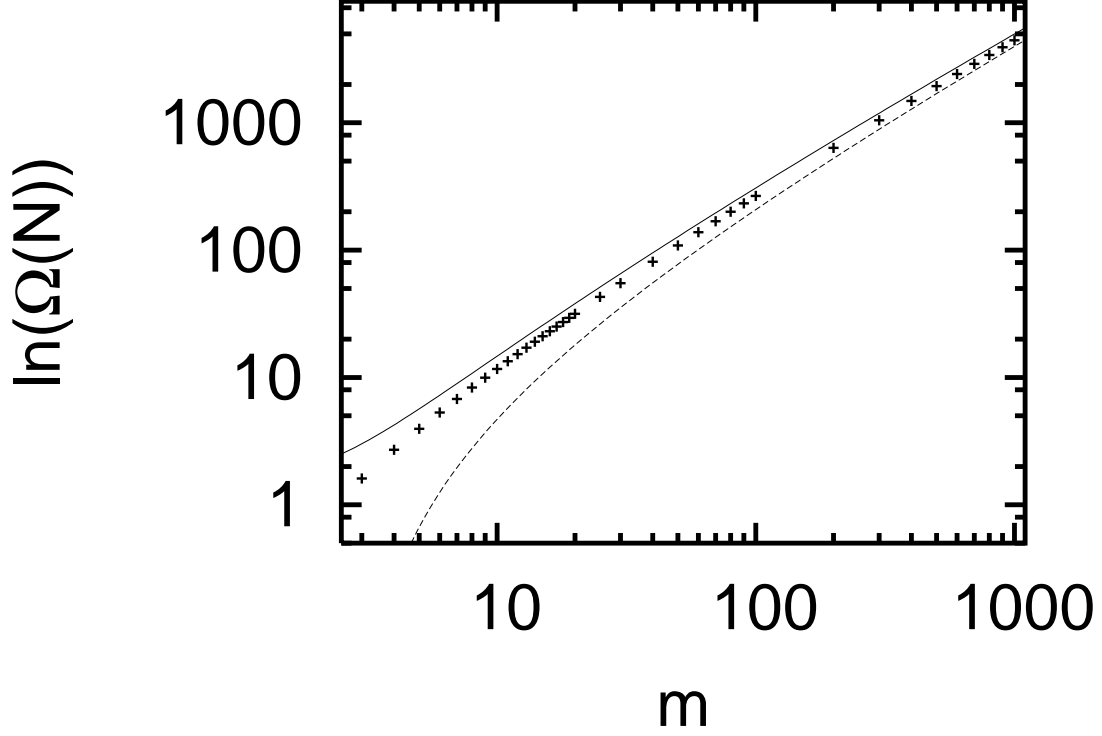


FIG. 7: Upper and lower bound (C11) for the logarithm of the number of factorisations of large integers  $N$  which are the products of the first  $m$  primes, in comparison with exact numerical data (crosses). It follows that an asymptotic upper bound on  $\ln \Omega(N)$  behaves at least as  $\ln N$ .

numbers (17) with different, *i.e.*, distinguishable prime factors, the probability distribution (7), when normalised to unit variance, approaches a Gaussian for  $N \rightarrow \infty$ . In contrast, if the prime factors are taken to be equal, *i.e.*, indistinguishable, the distribution remains distinctly non-Gaussian even in the asymptotic limit. Thus, we encounter here a fairly nontrivial model for the occurrence of non-Gaussian statistics.

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## APPENDIX A: EIGENVALUES FOR A LOGARITHMIC POTENTIAL

Let us consider the motion of a particle with mass  $m$  in a one-dimensional potential

$$V(x) = V_0 \ln\left(\frac{|x|}{L}\right), \quad (\text{A1})$$

where  $V_0$  and  $L$  are positive constants with the dimension of energy and length, respectively. Within the quasi-classical Bohr-Sommerfeld approximation<sup>26</sup>, the quantum mechanical energy eigenvalues  $\varepsilon_\nu$  pertaining to this potential are obtained by setting the classical action

$$I = \frac{1}{2\pi} \oint p \, dx \quad (\text{A2})$$

equal to  $\hbar(\nu + 1/2)$ , where  $p = p(x)$  denotes the classical momentum at the position  $x$ , and  $\nu = 0, 1, 2, \dots$  is an integer.

Introducing the right turning point  $x_\nu$  corresponding to classical motion with energy  $\varepsilon_\nu$ ,

$$x_\nu = L \exp\left(\frac{\varepsilon_\nu}{V_0}\right), \quad (\text{A3})$$

and accounting for the symmetry of the potential, the quantisation condition becomes

$$\begin{aligned} \frac{\hbar\pi}{2} \left(\nu + \frac{1}{2}\right) &= \int_0^{x_\nu} dx \sqrt{2m \left(\varepsilon_\nu - V_0 \ln \frac{x}{L}\right)} \\ &= \sqrt{2mV_0} \int_0^{x_\nu} dx \sqrt{-\ln \frac{x}{x_\nu}} \\ &= \sqrt{2mV_0} x_\nu \Gamma\left(\frac{3}{2}\right), \end{aligned} \quad (\text{A4})$$

giving

$$x_\nu = \sqrt{\frac{\pi}{2mV_0}} \hbar \left(\nu + \frac{1}{2}\right). \quad (\text{A5})$$

Utilising the relation (A3) for expressing the turning point through the energy, this latter equation yields the desired approximate eigenvalues

$$\frac{\varepsilon_\nu}{V_0} = \ln(2\nu + 1) + \ln\left(\frac{\hbar}{2L} \sqrt{\frac{\pi}{2mV_0}}\right). \quad (\text{A6})$$

## APPENDIX B: DERIVATION OF THE RECURSION RELATION

The derivation of the recursion relation (9) presented here is not restricted to logarithmic energy levels and thus generalises the derivation previously given in ref.<sup>27</sup>.

As in section II, let  $\Gamma_k(N)$  be the number of possible factorisations of  $N$  into  $k$  or less natural numbers larger than 1. If there are less than  $k$  factors, say  $k - m$ , we multiply the product by  $1^m$ . (This emphasises the correspondence with

the Bose condensate: The factor  $1^m$  represents  $m$  particles which reside in the ground state, carrying no energy.) For example,  $\Gamma_4(12)$  gives rise to the four factorisations

$$\begin{aligned} 12 &= 2 \cdot 2 \cdot 3 \cdot 1 \\ &= 2 \cdot 6 \cdot 1 \cdot 1 \\ &= 3 \cdot 4 \cdot 1 \cdot 1 \\ &= 12 \cdot 1 \cdot 1 \cdot 1 . \end{aligned} \tag{B1}$$

Keeping both  $N$  and  $k$  fixed, and randomly selecting one of the possible factorisations, the probability for the factor  $\nu$  to occur at least  $n$  times is given by

$$P_\nu^{\geq}(n) = \begin{cases} \frac{\Gamma_{k-n}(N/\nu^n)}{\Gamma_k(N)} & : N \bmod \nu^n = 0 \\ 0 & : \text{else} \end{cases} . \tag{B2}$$

The probability to find the factor  $\nu$  exactly  $n$  times is then obtained as a difference,

$$P_\nu(n) = P_\nu^{\geq}(n) - P_\nu^{\geq}(n+1) . \tag{B3}$$

Next, let  $\#_\nu$  be the number of occurrences of the factor  $\nu$  in some product. Taking the average over all possible products, we obtain

$$\begin{aligned} \overline{\#_\nu} &\equiv \sum_{n=1}^k n P_\nu(n) \\ &= \sum_{n=1}^k P_\nu^{\geq}(n) \\ &= \frac{1}{\Gamma_k(N)} \sum_{\substack{n=1 \\ N \bmod \nu^n = 0}}^k \Gamma_{k-n}(N/\nu^n) . \end{aligned} \tag{B4}$$

Since there is no possibility to factorise  $N > 1$  such that zero factors are larger than 1, we have  $\Gamma_0(N > 1) = 0$ ; since, however, there trivially is such a possibility for  $N = 1$ , it follows that  $\Gamma_0(1) = 1$ .

By definition of  $\#_\nu$  we have for every factorisation

$$k = \sum_{\nu=1}^N \#_\nu , \tag{B5}$$

which also holds upon averaging ( $k = \sum_{\nu=1}^N \overline{\#_\nu}$ ). Hence,

$$k = \sum_{\nu=1}^N \frac{1}{\Gamma_k(N)} \sum_{\substack{n=1 \\ N \bmod \nu^n = 0}}^k \Gamma_{k-n}(N/\nu^n) , \tag{B6}$$

leading immediately to

$$\Gamma_k(N) = \frac{1}{k} \sum_{\nu=1}^N \sum_{\substack{n=1 \\ N \bmod \nu^n = 0}}^k \Gamma_{k-n}(N/\nu^n) . \tag{B7}$$

Since the sums are finite, we can safely exchange the order of summation and arrive at the recursion relation (9).

### APPENDIX C: ASYMPTOTICS FOR PRODUCTS OF DISTINCT PRIMES

With an obvious simplification of notation, the recursion relation (18) for the number of ways to decompose a product of  $m$  distinct primes into exactly  $k$  integer factors takes the form

$$\Phi_{m,k} = k\Phi_{m-1,k} + \Phi_{m-1,k-1} ; \quad \Phi_{m,1} = 1 , \quad \Phi_{m,k>m} = 0 . \tag{C1}$$

We will first show by induction over  $k$  that the solution to this relation is given by

$$\Phi_{m,k} = \frac{(-1)^k}{k!} \sum_{\ell=1}^k (-1)^\ell \binom{k}{\ell} \ell^m. \quad (\text{C2})$$

For  $k = 1$ , equation (C2) obviously is correct. Moreover, we have

$$\begin{aligned} & k\Phi_{m-1,k} + \Phi_{m-1,k-1} \\ &= k \frac{(-1)^k}{k!} \sum_{\ell=1}^k (-1)^\ell \binom{k}{\ell} \ell^{m-1} + \frac{(-1)^{k-1}}{(k-1)!} \sum_{\ell=1}^{k-1} (-1)^\ell \binom{k-1}{\ell} \ell^{m-1} \\ &= \frac{(-1)^k}{k!} \left\{ \sum_{\ell=1}^k (-1)^\ell k \binom{k}{\ell} \ell^{m-1} - \sum_{\ell=1}^{k-1} (-1)^\ell k \binom{k-1}{\ell} \ell^{m-1} \right\}. \end{aligned} \quad (\text{C3})$$

Using

$$\binom{k-1}{\ell} = \frac{k-\ell}{k} \binom{k}{\ell}, \quad (\text{C4})$$

this yields

$$\begin{aligned} & k\Phi_{m-1,k} + \Phi_{m-1,k-1} \\ &= \frac{(-1)^k}{k!} \left\{ \sum_{\ell=1}^k (-1)^\ell k \binom{k}{\ell} \ell^{m-1} - \sum_{\ell=1}^{k-1} (-1)^\ell (k-\ell) \binom{k}{\ell} \ell^{m-1} \right\} \\ &= \frac{(-1)^k}{k!} \left\{ (-1)^k k \binom{k}{k} k^{m-1} + \sum_{\ell=1}^{k-1} (-1)^\ell \ell \binom{k}{\ell} \ell^{m-1} \right\} \\ &= \frac{(-1)^k}{k!} \sum_{\ell=1}^k (-1)^\ell \binom{k}{\ell} \ell^m, \end{aligned} \quad (\text{C5})$$

which proves the assertion (C2).

This result can now be employed to estimate the total number  $\Omega(\prod_{i=1}^m p_i)$  of factorisations for numbers (17) containing no identical prime factors:

$$\begin{aligned} \Omega\left(\prod_{i=1}^m p_i\right) &= \sum_{k=1}^m \Phi_{m,k} \\ &= \sum_{k=1}^m \sum_{\ell=1}^k \frac{(-1)^{k-\ell}}{(k-\ell)!} \frac{\ell^m}{\ell!} \\ &= \sum_{\ell=1}^m \left( \sum_{\nu=0}^{m-\ell} \frac{(-1)^\nu}{\nu!} \right) \frac{\ell^m}{\ell!} \\ &= \sum_{\ell=1}^m \frac{\Gamma(m-\ell+1, -1)}{e(m-\ell)!} \frac{\ell^m}{\ell!}, \end{aligned} \quad (\text{C6})$$

where  $\Gamma(a, b)$  denotes the incomplete Gamma function. The last equality here is proven by induction, using the recursive definition of  $\Gamma(m, -1)$ :

$$\Gamma(m+1, -1) = (-1)^m e + m \Gamma(m, -1), \quad \Gamma(1, -1) = e. \quad (\text{C7})$$

For large  $m$ , the sum (C6) is dominated by terms with  $m-\ell+1 \gg 1$ , so that we may use the asymptotic relation  $\Gamma(m-\ell+1, -1) = \int_{-1}^{\infty} t^{m-\ell} e^{-t} dt \sim \Gamma(m-\ell+1)$ . This leads to the asymptotic equality

$$\begin{aligned} \Omega\left(\prod_{i=1}^m p_i\right) &\sim \frac{1}{e} \sum_{\ell=1}^m \frac{\ell^m}{\ell!} \\ &\sim \frac{1}{e} \int_1^{m+1} dx \exp[f(x)], \end{aligned} \quad (\text{C8})$$



where, according to Stirling's formula,

$$f(x) \sim m \ln x - x \ln x + x . \quad (\text{C9})$$

The saddle-point approximation to this latter integral (C8) then gives

$$\Omega \left( \prod_{i=1}^m p_i \right) \sim \sqrt{2\pi} \exp \left[ \left( m + \frac{1}{2} \right) \ln m - m + \frac{m}{\ln m} - \ln(\ln m) - 1 \right] , \quad (\text{C10})$$

resulting in the asymptotic bounds

$$m \ln m - m < \ln \left[ \Omega \left( \prod_{i=1}^m p_i \right) \right] < m \ln m . \quad (\text{C11})$$